

Coherent states of quantum non-linear systems

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Abstract

Quantum dynamics of integrable systems is discussed. Localized wave packets generalizing the conventional coherent states of minimal uncertainty are constructed. The wave packet moves along a certain trajectory and does not change its shape for times of order $\frac{1}{\hbar}$.

In this Letter we suggest that the definition of a coherent state should be related to the Hamiltonian. The conventional coherent states [1] are defined as eigenstates of the annihilation operator

$$\alpha |z\rangle = z |z\rangle \quad (1)$$

The real and the imaginary parts of the complex variable z can be expressed by the mean values of the position \hat{x} and momentum \hat{p} operators in the coherent state. Under the oscillator Hamiltonian evolution U_t

$$U_t |z\rangle = |\exp(-i\omega t)z\rangle \quad (2)$$

where ω is the oscillator's frequency. Then, because the state $|z\rangle$ is concentrated around $x = \sqrt{\frac{2\hbar}{\omega}} \operatorname{Re} z$ it follows that $U_t |z\rangle$ has its support on the classical trajectory $x(t) = \sqrt{\frac{2\hbar}{\omega}} \operatorname{Re}(\exp(-i\omega t)z)$. The simple classical evolution $z \rightarrow \exp(-i\omega t)z$ is a consequence of the simple representation of $\sqrt{\omega}x + i\frac{p}{\sqrt{\omega}} = \sqrt{I} \exp(i\theta)$ in terms of the action-angle variables (I, θ) for the harmonic oscillator.

We are going to generalize the construction of coherent states to integrable non-linear systems. It is known that the quantum non-linear dynamics cannot be transformed into a coherent state dynamics exactly (see the discussion in [2][3]; these authors define generalized coherent states, which do not coincide with ours). However, wave packets approximately localized around the classical trajectory of an electron in the hydrogen atom have been constructed [4][5]. For this purpose a large dynamical symmetry relating the hydrogen atom to the fourdimensional oscillator has been utilized. In this Letter we do not assume any symmetry, but the integrability of a classical Hamiltonian system. Our construction of wave packets can be applied to multidimensional anharmonic

oscillators. The localized states can describe a classical behavior of quantum molecules [6].

The wave packets are sums over energy eigenstates $\psi_r(x)$ ($x \in R^n$; here and later on we avoid the vector notation and omit vector indices if there is no danger of confusion). For anharmonic oscillators the leading behavior for large $|x|$ of $\psi_r(x)$ is determined by the behavior of the ground state χ [7]. We write the ground state in the form $\chi \equiv \exp(-\frac{S}{\hbar})$. It is the solution of the Hamiltonian eigenvalue problem with the eigenvalue E_q

$$H\chi \equiv (-\frac{\hbar^2}{2m}\Delta + V)\chi = E_q\chi \quad (3)$$

Let us make the similarity transformation

$$\check{H} \equiv \hbar^{-1}\chi^{-1}H\chi = -\frac{\hbar}{2m}\Delta + \frac{1}{m}\nabla S\nabla \quad (4)$$

Then, \check{H} has a formal limit when $\hbar \rightarrow 0$ equal to $\frac{1}{m}\nabla S\nabla$ (note that S is real if we restrict ourselves to systems with a non-degenerate ground state χ).

Let \mathcal{P} be an analytic function on the configuration space. Let us consider the Heisenberg equations of motion on a state $\exp(-\frac{S}{\hbar})\mathcal{P}$ (we mark position and momentum operators by hats in order to distinguish them from an argument of a wave function)

$$-m\frac{d\hat{x}_t}{dt}\exp(-\frac{S}{\hbar})\mathcal{P} = \hat{p}_t\exp(-\frac{S}{\hbar})\mathcal{P} \quad (5)$$

where

$$\hat{x}_t = U_t\hat{x}U_t^\dagger \quad (6)$$

Using eq.(4) we can rewrite eq.(5) in the form

$$-m\frac{d\hat{x}_t}{dt}\exp(-\frac{S}{\hbar})\mathcal{P} = \exp(-\frac{S}{\hbar})\check{U}_t(-i\hbar\nabla + i\nabla S)\check{U}_t^\dagger\mathcal{P} \quad (7)$$

where \check{U}_t is the unitary group generated by \check{H} (eq.(4)). The limit $\hbar \rightarrow 0$ in eq.(7) can be obtained explicitly. This is so because $\hbar\nabla$ acting on an analytic function of \hbar gives a function of order \hbar . Then, in the limit $\hbar \rightarrow 0$

$$\check{U}_t\mathcal{P}(x) = \mathcal{P}(x_t(x)) + O(\hbar) \quad (8)$$

where $x_t(x)$ is the solution of the c-number equation

$$m\frac{dx_t}{dt} = -i\nabla S(x_t) \quad (9)$$

with the initial condition x .

Neglecting terms of order \hbar we can see that equation (7) is simplified to

$$-m\frac{d\hat{x}_t}{dt}\exp(-\frac{S}{\hbar})\mathcal{P} = \exp(-\frac{S}{\hbar})(i\nabla S(x_t))\mathcal{P}$$

Hence, by iteration we obtain the equality

$$f(\hat{x}_t)\mathcal{P}exp(-\frac{S}{\hbar}) = f(x_t)\mathcal{P}exp(-\frac{S}{\hbar}) \quad (10)$$

for an arbitrary analytic function f , where x_t (without the hat) is the solution of the c-number differential equation (9).

The definition of the coherent state $|z\rangle$ is now determined by the requirements

- i) $|z=0\rangle = \chi$
- ii) $|z\rangle$ is transformed into a certain $|z(t)\rangle$ under the quantum Hamiltonian evolution (8) in the limit $\hbar \rightarrow 0$.

For the construction of $|z\rangle$ let us consider a solution $S_{cl}^I(x)$ of the "imaginary" Hamilton-Jacobi equation (outside the classical domain)

$$-\frac{1}{2m}(\nabla S_{cl}^I)^2 + V = E_{cl}(I) \quad (11)$$

depending on n parameters I_k which can be related to the action variables. We assume that $E_{cl} \equiv E_{cl}(I=0)$ is the classical ground state energy. It is known that $E_q - E_{cl} \simeq O(\hbar)$. We denote by S_{cl} the solution of eq.(11) with the ground state energy E_{cl} . The difference between S and S_{cl} is negligible in the limit $\hbar \rightarrow 0$. This can be seen from equation (3) which when expressed by S reads

$$-\frac{1}{2m}(\nabla S)^2 + \frac{\hbar}{2m}\Delta S + V = E_q \quad (12)$$

Subtracting eqs.(11) and (12) we can conclude that $S - S_{cl} \simeq O(\hbar)$.

The search for a state $|z\rangle$ fulfilling our assumptions i)-ii) is simplified if we find operators Q_k which under the quantum evolution $exp(-it\hat{H})$ transform multiplicatively

$$Q_k|\psi\rangle \rightarrow \lambda_k(t)Q_k|\psi\rangle \quad (13)$$

on a dense set of vectors $|\psi\rangle$. We find that the proper guess is

$$Q_k = exp(\frac{\partial S_{cl}^I}{\partial I_k})|_{I=0} \quad (14)$$

From eq.(8) and the estimate on $S - S_{cl}$ it follows that \hat{H} in the limit $\hbar \rightarrow 0$ generates the complex classical evolution

$$\frac{d\xi}{dt} = -\frac{i}{m}\nabla S_{cl}(\xi) \quad (15)$$

We show that in this limit the variables $Q_k(t)$ fulfil linear equations. In fact, from eq.(14)

$$\frac{dQ_k}{dt}(\xi_t) = \frac{-i}{m} \frac{\partial^2 S_{cl}}{\partial I_k \partial \xi_r} \frac{\partial S_{cl}}{\partial \xi_r} Q_k(\xi_t) \quad (16)$$

Then, from eq.(11) it follows that

$$-\frac{1}{m} \frac{\partial^2 S_{cl}}{\partial x_r \partial I_s} \frac{\partial S_{cl}}{\partial x_r} = \frac{\partial E_{cl}}{\partial I_s} \quad (17)$$

Hence, setting $I = 0$ in eq.(17)

$$\frac{dQ_k}{dt}(\xi) = -i\omega_k Q_k(\xi) \quad (18)$$

where

$$\omega_k = \frac{\partial E_{cl}}{\partial I_k}(I = 0) \quad (19)$$

It follows that in eq.(13) $\lambda_k(t) = \exp(-i\omega_k t)$.

Now, in terms of the variables Q_k our definition of a coherent wave packet reads

$$\langle x | z \rangle \equiv \exp\left(-\frac{S}{\hbar} + \sqrt{\frac{2}{\hbar}} \sum_{k=1}^n \sqrt{\omega_k} z_k Q_k\right) \quad (20)$$

The linear dependence on Q of the exponential in eq.(20) has as a consequence that $|z\rangle$ is an eigenfunction of a generalized annihilation operator

$$\alpha_k = \sqrt{\frac{\hbar}{2\omega_k}} \frac{\partial}{\partial Q_k} + \frac{1}{\sqrt{2\hbar\omega_k}} \frac{\partial S}{\partial Q_k} \quad (21)$$

Namely

$$\alpha_k |z\rangle = z_k |z\rangle$$

It follows that

$$\langle z | \frac{1}{2}(\alpha_k + \alpha_k^\dagger) | z \rangle \langle z | z \rangle^{-1} = \text{Re} z_k \quad (22)$$

and

$$\langle z | \frac{1}{2i}(\alpha_k - \alpha_k^\dagger) | z \rangle \langle z | z \rangle^{-1} = \text{Im} z_k \quad (23)$$

In this way we obtain an interpretation of z as a mean value of a certain observable. Furthermore, the eigenvalue condition is a necessary condition for a minimal uncertainty property of the operators which constitute the real and imaginary part of α (see refs.[9] and [10]). It follows from eqs.(8),(18) and (20) that under the quantum evolution

$$U_t |z\rangle = |\exp(-i\omega t)z\rangle + O(\hbar)$$

There remains to investigate the localization properties of $|z\rangle$. The probability density $|\langle x | z \rangle|^2$ is maximal at a point Q determined by the equation

$$\frac{\partial S}{\partial Q_r} = \sqrt{2\hbar\omega_r} \text{Re} z_r \quad (24)$$

Eq.(24) gives an interpretation of Rez which coincides with the one of eq.(22) in the leading order of \hbar . The probability density $|\langle x|U_t|z\rangle|^2$ is concentrated on the solution of the equation

$$\frac{\partial S}{\partial Q_r} = \sqrt{2\hbar\omega_r} \text{Re}(\exp(-i\omega_r t)z_r) \quad (25)$$

It can be seen from the definition (14) of Q that $Q \sim x$ and $S(Q) \sim Q^2$ for the anharmonic oscillators in the weak coupling regime. So, in this sense the probability density $|\langle x|U_t|z\rangle|^2$ is concentrated on a deformed torus filled by the trajectories of the integrable system.

Let us consider two examples. The first concerns a onedimensional non-negative potential V normalized in such a way that $E_{cl} = 0$ in eq.(11). Then

$$S_{cl}(x) = \sqrt{2m} \int^x \sqrt{V(y)} dy \quad (26)$$

is uniquely defined (no problem with the square root). Hence

$$Q = \exp(\omega\sqrt{m} \int^x \frac{1}{\sqrt{2V(y)}} dy) \quad (27)$$

To be more specific let

$$V(y) = \frac{m\omega^2}{2} y^2 + gy^4 \quad (28)$$

We calculate the integral (26)

$$S_{cl}(x) = m\omega \int^x dy y \sqrt{1 + cy^2} = \frac{m\omega}{3c} (1 + cx^2)^{\frac{3}{2}} \quad (29)$$

where $c = \frac{2g}{m\omega^2}$. Now,

$$Q = \exp\left(\int^x dy (y \sqrt{1 + cy^2})^{-1}\right) = 2x(1 + \sqrt{1 + cx^2})^{-1} \quad (30)$$

As a second example let us consider a non-negative twodimensional rotationally symmetric potential $V(r)$. Then, the classical Hamilton-Jacobi function W reads (in the plane coordinates (r, ϕ))

$$W(r, \phi) = I_1 \phi + \int^r d\rho (2mE_{cl}(I_1, I_2) - 2mV(\rho) - I_1^2 \rho^{-2})^{\frac{1}{2}} \quad (31)$$

A solution of the "imaginary" Hamilton-Jacobi equation (11) is obtained in the limit $E_{cl}(I_1, I_2) \rightarrow 0$. For a non-negative potential this limit means $I_1 \rightarrow 0$ and $I_2 \rightarrow 0$. It follows that

$$S_{cl}(r, \phi) = \int^r d\rho (2mV(\rho))^{\frac{1}{2}} \quad (32)$$

and

$$Q_1 = \exp(i\phi + \omega_1 u(r)) \quad (33)$$

$$Q_2 = \exp(2\omega_2 u(r))$$

where

$$u(r) = \int^r d\rho (2mV(\rho))^{-\frac{1}{2}} \quad (34)$$

and ω_k are defined in eq.(19).

So far we have restricted ourselves to the limit $\hbar \rightarrow 0$. The corrections to the formulae (8)-(10) and (13) are discussed in our papers [11][12]. In particular, in ref.[12] we bound the growth in time of the $O(\hbar)$ term in eq.(8) by $\sqrt{\hbar t}$. We specify also some conditions which ensure that the $O(\hbar)$ term is bounded uniformly in time. On the basis of these results we estimate that the wave packet moves along the trajectory (25) and does not change its shape at least for a time of order $\frac{1}{\hbar}$. In general, it is known [6] (see also a rigorous formulation in ref.[8]) that a particle in a state belonging to the subspace of the Hilbert space corresponding to the discrete spectrum remains localized in a bounded region for an arbitrarily large time. Moreover, from the quasiperiodicity it follows that the wave packet recovers its shape after a sufficiently large time (see an estimate on this time in ref.[6]). Our claim is that the coherent states defined in this Letter do not lose their shapes during a time interval large in comparison to the classical periods $2\pi\omega_k^{-1}$.

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